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LETTER TO THE EDITOR

The matrix quantum unitary Cayley–Klein groups

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Abstract. We define the matrix quantum unitary Cayley-Klein groups of the arbitrary dimensions by generalizing the R-matrix theory of the quantum groups to the case of the degenerate Hermitic forms. The corresponding quantum groups are obtained from the appropriate classical quantum groups by the contractions.

Faddeev *et al* [1] have developed the R-matrix theory of quantum groups and quantum algebras, which correspond to the (single-parameter) deformation of the classical algebras A_n, B_n, C_n and D_n. For each classical algebra we have built a set of Cayley–Klein algebras, which are obtained from the corresponding classical algebras by the contractions and analytical continuations [2–4]. This letter is devoted to the construction of the quantum matrix unitary Cayley–Klein groups. The R-matrix approach [1] and unified description of the Cayley–Klein groups [2] naturally combine to yield the desired groups. We start with the general theory of the quantum matrix unitary Cayley–Klein groups and finally we consider the two-dimensional case of these quantum groups.

Let the R_q matrix be as in the case of A_n (or $\mathrm{su}(n+1)$) [1], i.e.

$$R_q = q\sum_{k=0}^{n} e_{kk}\otimes e_{kk} + \sum_{\substack{k,m=0\\k\neq m}}^{n} e_{kk}\otimes e_{mm} + (q-q^{-1})\sum_{\substack{k,m=0\\k>m}}^{n} e_{km}\otimes e_{mk}$$

$$(e_{km})_{ij} = \delta_{ik}\delta_{jm} \qquad k,m,i,j=0,1,\ldots,n. \tag{1}$$

Let us define the algebra $A_q(j)$ as the associative C-algebra generated by the non-commutative elements

$$(T(j))_{km} = t_{km} \quad k\geqslant m \qquad (T(j))_{km} = t_{km}J_{km}^2 \quad k<m$$

$$J_{km}=1 \quad k\geqslant m \qquad J_{km}=\prod_{r=k+1}^{m} j_r,\ k<m \quad j_r=1,\iota_r,i \quad r=1,2,\ldots,n \tag{2}$$

where ι_r are the dual numbers, which are nilpotent $\iota_r^2=0$ and obey the commutative laws of multiplication $\iota_r\iota_m=\iota_r\iota_m\neq 0$, $m\neq r$, [2, 3], and factorized by the following relationships:

$$R_qT_1(j)T_2(j)=T_2(j)T_1(j)R_q \tag{3}$$

where $T_1(j)=T(j)\otimes I$, $T_2(j)=I\otimes T(j)$. The algebra $A_q(j)$ is called the algebra of functions on the quantum group $\mathrm{GL}_q(n+1;j)$. The matrix $T(j)$ (2) is a multiplicative matrix whose entries generate $A_q(j)$ and therefore one has here the matrix quantum group $\mathrm{GL}_q(n+1;j)$. If one introduces the quantum determinant $\det_q T(j)$ as usual

$$\det_q T(j) = \sum_{\sigma\in\mathrm{Symm}(n+1)} (-q)^{l(\sigma)} J_{0\sigma_0}^2 t_{0\sigma_0} J_{1\sigma_1}^2 t_{1\sigma_1}\ldots J_{n\sigma_n}^2 t_{n\sigma_n} \tag{4}$$

where $l(\sigma)$ is the parity of the substitution σ, and factorizes $A_q(j)$ by the relationship $\det_q T(j)=1$, then one obtains the algebra $\mathrm{Fun}(\mathrm{SL}_q(n+1;j))$ of functions on the quantum group $\mathrm{SL}_q(n+1;j)$ and the matrix quantum group $\mathrm{SL}_q(n+1;j)$.

The associative algebra generated by $n+1$ generators $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n$ divided by the ideal corresponding to the relations $\tilde{x}_k\tilde{x}_m = q\tilde{x}_m\tilde{x}_k, 0 \leqslant k < m \leqslant n$ is called the $(n+1)$-dimensional quantum vector space $C_{q,n+1}$. The map

$$\psi : C_{q,n+1} \mapsto C_{q,n+1}(j) \qquad x = \Psi(j)\tilde{x} \qquad \Psi(j) = \mathrm{diag}(1, J_{01}, \ldots, J_{0n}) \tag{5}$$

is in accordance with the following transformations of the quantum vector Cayley-Klein space $C_{q,n+1}(j)$ by the matrix quantum group $\mathrm{SL}_q(n+1;j)$

$$\delta(x) = T(j)\dot{\otimes} x \qquad \delta(x_k) = \sum_{m=0}^{k} t_{km} \otimes x_m + \sum_{m=k+1}^{n} J_{km}^2 t_{km} \otimes x_m. \tag{6}$$

These transformations represent the homomorphisms of quantum spaces.

Let deformation parameter $q \in R$. The matrix quantum unitary Cayley-Klein group $\mathrm{SU}_q(n+1;j)$ is defined with help of the involution $*$:

$$T^{t}(j)\Psi^2(j)T^*(j) = \Psi^2(j) \tag{7}$$

where T^t is the transpose matrix and transforms the quantum Hermitic Cayley-Klein space

$$U_{q,n+1}(j) = \{x_0, \ldots, x_n, y_0, \ldots, y_n \,|\, x_k x_m = q x_m x_k,\ y_k y_m = q^{-1} y_m y_k,\ k<m,\ y_k^* = x_k\} \tag{8}$$

due to the following equations:

$$\delta(x) = T(j)\dot{\otimes} x \qquad \delta(y) = S(T(j))^t \dot{\otimes} y \tag{9}$$

where S is an antipode

$$S(J_{km}^2 t_{km}) = (-q)^{k-m}\tilde{t}_{mk}(j)$$

$$\tilde{t}_{mk}(j) = \sum_{\sigma \in \mathrm{Symm}(n)} (-q)^{l(\sigma)} J_{0\sigma_0}^2 t_{0\sigma_0} \ldots J_{m-1,\sigma_{m-1}}^2 t_{m-1,\sigma_{m-1}} J_{m+1,\sigma_{m+1}}^2 t_{m+1,\sigma_{m+1}} \ldots J_{n\sigma_n}^2 t_{n\sigma_n} \tag{10}$$

$$\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{m-1}, \sigma_{m+1}, \ldots, \sigma_n) = \sigma(0, 1, \ldots, k-1, k+1, \ldots, n).$$

This antipode matrix $S(T(j))$ obeys the usual property

$$T(j)S(T(j)) = S(T(j))T(j) = I. \tag{11}$$

The transformations (9) keep invariant the quadratic form

$$x^{*t}\Psi^2(j)x = x_0^* x_0 + \sum_{k=1}^{n} J_{0k}^2 x_k^* x_k. \tag{12}$$

The algebra $\mathrm{Fun}(\mathrm{SL}_q(n+1;j))$ with the involution (7) is denoted by $\mathrm{Fun}(\mathrm{SU}_q(n+1;j))$ and is called the algebra of functions on the quantum matrix unitary Cayley-Klein group $\mathrm{SU}_q(n+1;j)$. If one introduces to the algebra $\mathrm{Fun}(\mathrm{SU}_q(n+1;j))$ the coproduct

Δ and counit ε as follows:

$$\Delta(T(j)) = T(j)\dot{\otimes} T(j) \qquad \Delta(I) = I\otimes I$$
$$J^2_{km}\Delta(t_{km}) = \sum_{p=0}^{n} J^2_{kp}J^2_{pm} t_{kp}\otimes t_{pm} \qquad \varepsilon(T(j)) = I \tag{13}$$

then with the antipode (10) we obtain the Hopf algebra structure.

If all parameters j are equal to real unit $j_1 = \ldots j_n = 1$, then we are in one-to-one correspondence with R-matrix theory of the quantum unitary groups [1]. The pseudo-Hermitic cases $j_k = 1, i$ $(k = 1, \ldots, n)$ with different signatures have also been regarded by Faddeev *et al* [1]. But when some parameters j take the *dual* values, then the matrix $\Psi^2(j)$ and Hermitic form $x^{*t}\Psi^2(j)x$ are *degenerate*. Such cases have not been considered in the literature [1, 5]. The corresponding quantum matrix unitary Cayley–Klein group $\mathrm{SU}_q(n+1; j)$ is obtained from the classical quantum matrix unitary group $\mathrm{SU}_q(n+1)$ by (multidimensional) *contractions*.

For the simplest two-dimensional case the R_q matrix is in the form

$$R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{14}$$

and does not depend on the parameter j_1. This R_q matrix is the same for all three quantum matrix Cayley–Klein groups $\mathrm{SU}_q(2; j_1), j_1 = 1, \iota_1, i$. (The corresponding quantum algebras have been considered in [6]). Also we do not transform the quantum parameter q under contractions.

The generating matrix $T(j_1)$ with non-commuting elements is as follows:

$$T(j_1) = \begin{pmatrix} t_{00} & j_1^2 t_{01} \\ t_{10} & t_{11} \end{pmatrix}. \tag{15}$$

The commutation relations for their matrix elements follow from equation (3) in the form

$$t_{10}t_{11} = qt_{11}t_{10} \qquad j_1^2 t_{00}t_{01} = j_1^2 q t_{01}t_{00} \qquad j_1^2 t_{01}t_{11} = j_1^2 q t_{11}t_{01} \qquad j_1^2 t_{01}t_{10} = j_1^2 t_{10}t_{01}$$
$$t_{00}t_{10} = qt_{10}t_{00} \qquad t_{00}t_{11} - j_1^2 q t_{01}t_{10} = t_{11}t_{00} - j_1^2 q^{-1} t_{01}t_{10} \equiv \det_q T(j_1) = 1. \tag{16}$$

The antipode of $T(j_1)$ is easily obtained and is given by the matrix

$$S(T(j_1)) = \begin{pmatrix} t_{11} & -j_1^2 q^{-1} t_{01} \\ -qt_{10} & t_{00} \end{pmatrix}. \tag{17}$$

For real q the equation (7), with $T(j_1)$ as in equation (15) and the matrix $\Psi^2(j_1) = \mathrm{diag}(1, j_1^2)$, is now in the explicit form

$$\begin{pmatrix} t_{00}t_{00}^* + j_1^2 t_{10}t_{10}^* & j_1^2(t_{00}t_{01}^* + t_{10}t_{11}^*) \\ j_1^2(t_{01}t_{00}^* + t_{11}t_{10}^*) & j_1^2(t_{11}t_{11}^* + j_1^2 t_{01}t_{01}^*) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & j_1^2 \end{pmatrix}. \tag{18}$$

These equations define the involutive matrix $T^*(j_1)$, which for $j_1 \neq \iota_1$ is as follows:

$$T^*(j_1) = \begin{pmatrix} t_{11} & -j_1^2 q^{-1} t_{10} \\ -qt_{01} & t_{00} \end{pmatrix}. \tag{19}$$

Let us regard the contraction $j_1 = \iota_1$. Then we have the matrix

$$T(\iota_1) = \begin{pmatrix} t_{00} & 0 \\ t_{10} & t_{11} \end{pmatrix} \tag{20}$$

with the following commutation relations for their entries: $t_{00}t_{10} = qt_{10}t_{00}$, $t_{10}t_{11} = qt_{11}t_{10}$, $t_{00}t_{11} = t_{11}t_{00}$. From $\det_q T(\iota_1) = t_{00}t_{11} = 1$ follows $t_{11} = t_{00}^{-1}$ and from the matrix equation (18) we obtain $t_{00}t_{00}^* = 1$, $t_{11}t_{11}^* = 1$ or $t_{00}^* = t_{00}^{-1}$, $t_{11}^* = t_{11}^{-1} = t_{00}$. Therefore we can write $t_{00} = \exp(i\varphi_0)$, $t_{00}^* = \exp(-i\varphi_0)$, $t_{11} = \exp(-i\varphi_0)$, $t_{11}^* = \exp(i\varphi_0)$. The generator t_{10}^* is not expressed by the generators t_{km} and therefore is an independent generator.

The matrix $T(\iota_1)$, the antipode $S(T(\iota_1))$ and the involutive matrix $T^*(\iota_1)$ of the contracted quantum matrix unitary group $SU_q(2; \iota_1)$ now take the final forms

$$T(\iota_1) = \begin{pmatrix} e^{i\varphi_0} & 0 \\ t_{10} & e^{-i\varphi_0} \end{pmatrix} \qquad S(T(\iota_1)) = \begin{pmatrix} e^{-i\varphi_0} & 0 \\ -qt_{10} & e^{i\varphi_0} \end{pmatrix}$$
$$T^*(\iota_1) = \begin{pmatrix} e^{-i\varphi_0} & 0 \\ t_{10}^* & e^{i\varphi_0} \end{pmatrix} \tag{21}$$

with the commutation relations $\exp(i\varphi_0)t_{10} = qt_{10}\exp(i\varphi_0)$, $t_{10}^*\exp(i\varphi_0) = q\exp(i\varphi_0)t_{10}^*$. *It is noted that the involutive matrix* $T^*(\iota_1)$ *(21) is not obtained from the involutive* matrix $T^*(j_1)$ (19) for $j_1 = \iota_1$, because the equation (19) is valid only for $j_1 \neq \iota_1$.

We have generalized the R-matrix theory of the quantum unitary groups [1] to the case of the matrix quantum unitary Caley-Klein groups by defining the generating $T(j)$, the antipode $S(T(j))$ and the involutive $T^*(j)$ matrices. The main feature of the developed approach is that all the quantum Cayley-Klein groups have the same R_q matrix, i.e. the quantum (or deformation) parameter q is not transformed under *transitions from one group to another. As for contractions corresponding to the dual* values of the parameters j, we conclude that the dual numbers are the real tool to regard the contractions of groups, not only in the classical [2, 3], but also in the quantum case (see also [6], where the contractions of the quantum algebras $SU_q(2)$ and $SO_q(3)$ were investigated with the help of the dual units).

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